Lecture notes on risk management, public policy, and the financial system

Credit portfolios

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Overview of credit portfolio risk

Core difficulties in credit modeling

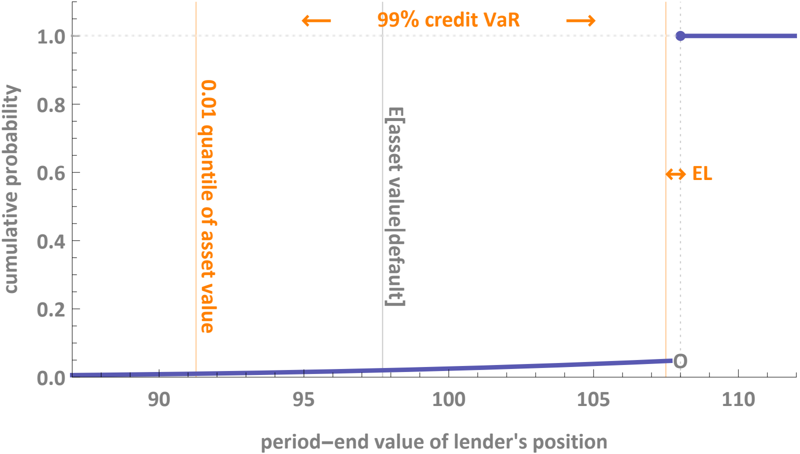
Sparse data: default infrequent, joint default even more infrequent

- In some years even spec grade realized default rate is zero

Skewness of credit risk: market risk may have fat tails, but generally "continuous" distributions

- Exception: currencies with fixed exchange rates
- Credit risk of single obligor and even portfolios closer to binary
- Senior structured credit closer to binary

Skewness of credit risk



Probability distribution of bond value one year hence in the Merton model. Firm assets' drift rate 10 percent, annual volatility 25 percent, and initial value 145; debt consists of a bond, par value 100 and 8 percent coupon. Default probability is 4.91 percent, so 95.09 percent of the probability mass is located at a single point.

Credit portfolio risk concepts

Default correlation: measure of the likelihood that 2 firms both default in the next year

- Default correlation is an *event* correlation$\leftrightarrow$asset return correlation
- Portfolio lender generally doesn't want even low-probability possibility of "cluster" of defaults
- Exception: ($\rightarrow$)structured product equity tranche

Granularity or diversification: many small debt obligations relative to total portfolio

- Often measured via **Herfindahl index**

Credit Value-at-Risk defined as

α-quantile of credit loss distribution minus EL

- Portfolio credit managers, banks, take account of expected losses in reserving, capital planning

Approaches to credit portfolio risk modeling

- Basic model types
  - Closed-form: single-factor model
  - Simulation: copula model

Joint default in a two-credit portfolio

- Simplest framework: two obligors (households, firms or countries)
  - Fixed time horizon τ years
  - Event of default Bernoulli distributed
  - τ-year probabilities of default of obligors 1 and 2 denoted π_1 and π_2
  - Joint default probability—probability both obligors default—denoted π_{12}
- Joint default distribution↔product of two (possible correlated) Bernoulli variates x_1 and x_2:

Outcome	x_1	x_2	$x_1 x_2$	Probability
Both firms default	1	1	1	π_{12}
Firm 1 only defaults	1	0	0	$\pi_1 - \pi_{12}$
Firm 2 only defaults	0	1	0	$\pi_2 - \pi_{12}$
No default	0	0	0	$1 - \pi_1 - \pi_2 + \pi_{12}$

Default correlation in a two-credit portfolio

- For any pair of credits, default correlation defined as rank correlation:

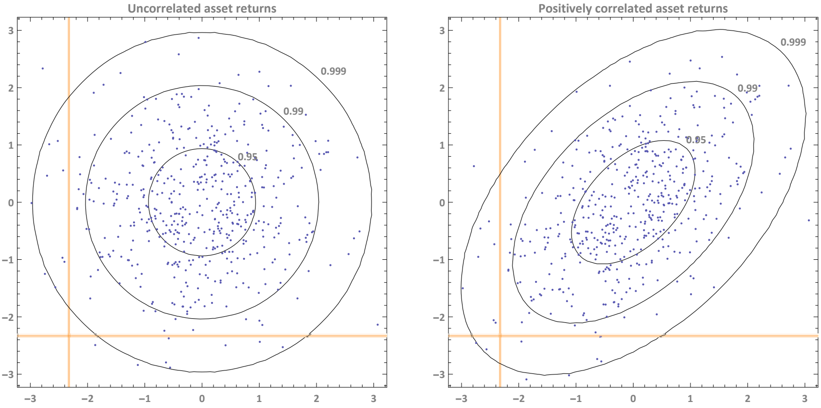
$$\boxed{\rho_{12} = \frac{\pi_{12} - \pi_1\pi_2}{\sqrt{\pi_1(1-\pi_1)}\sqrt{\pi_2(1-\pi_2)}}}$$

  - Default correlation is zero $\Leftrightarrow \pi_{12} = \pi_1\pi_2$
- Identical firms: if $\pi_1 = \pi_2 = \pi$, simplifies to:

$$\rho_{12} = \frac{\pi_{12} - \pi^2}{\pi(1-\pi)}$$

- Examples:
  - $\pi_1 = \pi_2 = 0.01$, $\pi_{12} = 0.0005$: $\rho_{12} = 0.040404$
  - $\pi_1 = \pi_2 = 0.10$, $\pi_{12} = 0.0250$: $\rho_{12} = 0.166667$
- Joint default probability and default correlation generally small numbers, since default infrequent

Correlated and uncorrelated defaults



Probability of joint default of two borrowers, each with default probability 1 percent. Left panel: zero correlation. Right panel: moderate positive correlation coefficient $\rho = 0.50$.

Default correlation and credit portfolio risk

- Key risk to capture: extreme credit events
- Default correlation related to default clustering and concept of ($\rightarrow$)**contagion** of financial distress/insolvency among firms
- Skewness and tail risk amplified by clusters of defaults and/or high loss given default (LGD)
  - Higher default correlation makes clusters of defaults likelier for wide range of default probabilities
  - Structuring/tranching can alter both clustering and LGD

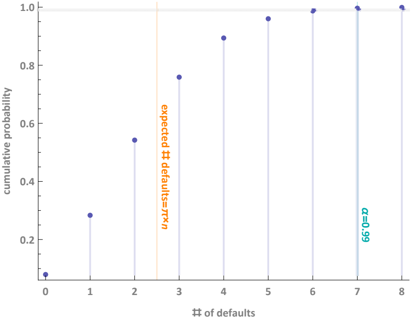
Credit analysis of an uncorrelated portfolio

- Simple credit portfolios with uncorrelated defaults
  - Portfolio of n identical loans or bonds
  - All pairwise default correlations zero
  - All default probabilities equal π
- $\Rightarrow$Number of defaults follows **binomial distribution** with parameters n and π
- Determine probability distribution of **number of defaults** or **default count**
  - **Expected number of defaults**—the expected value of the default count—is πn
  - Can compute probabilities and quantiles of the default count
- Then use loan par values to determine distribution of **credit loss**

Uncorrelated default count distribution: example

- Number of loans $n = 100$
- Default probability $\pi = 0.025$
- Default correlation zero
- 0.99-quantile of default count is 7
- Binomial distribution table:

# defaults	cumul. prob
0	0.0795
1	0.2834
2	0.5422
3	0.7590
4	0.8937
5	0.9601
6	0.9870
7	0.9963
8	0.9991



Cumulative probability function of number of defaults. Orange grid line at expected default count. Cyan grid line at 0.99-quantile of default count. The x-axis is truncated at 8 defaults.

Credit loss distribution in an uncorrelated portfolio

- With additional data on the *term* and *size* (par value) of the n loans, we can determine distribution of credit loss in currency units

  Credit loss: default count $\times$ loan size

  Expected loss: expected value of credit loss, default probability $\times$ portfolio total par value

  Credit Value-at-Risk: a high quantile of default count $\times$ loan size $-$ expected loss
- Simplifying assumptions
  - Set loan term equal to risk/VaR horizon
  - Default only at maturity$\Leftrightarrow$zero- or single-coupon loans
  - Recovery equal to zero$\Leftrightarrow$LGD 100 percent
  - Identical loans$\Leftrightarrow$loan size $= n^{-1}\times$ portfolio total par value
- **Example**: portfolio total par value \$1 000 000, $n = 100$, $\pi = 0.025$
  - Loan size \$10 000

	$\alpha = 0.95$	$\alpha = 0.99$
Loss quantile (no. loans)	5	7
Loss quantile (\$)	50 000	70 000
Credit VaR (\$)	25 000	45 000

Granularity reduces risk

- Higher granularity reduces default loss variance, turns expected default loss into a "cost"
- Effect is greatest for low default probabilities
- Risk reduction effect of granularity is much lower in a portfolio with high correlation
  - For example, granular mortgage pool, but regionally concentrated and with high-risk borrowers
- High granularity similar in economic effect to low default correlation and v.v.
  - Low granularity→very large losses with low but material probability

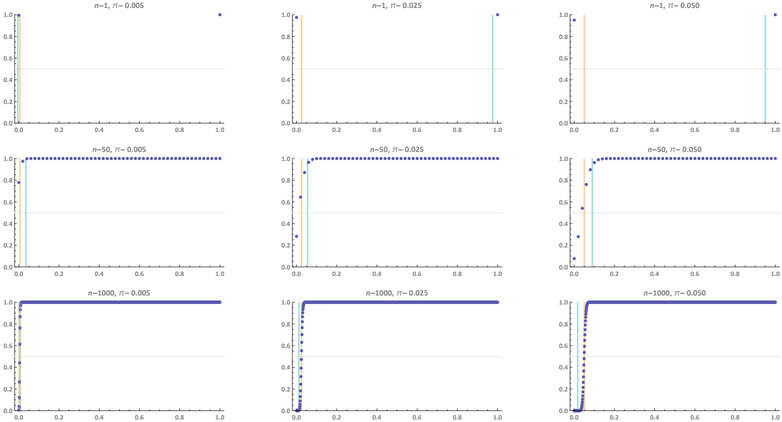
Granularity and risk: example

- Parameters of the example:
  - $n = \{1, 50, 1000\}$ one-year zero-coupon loans
  - $\pi = \{0.005, 0.025, 0.05\}$ default probability
  - Default correlation set to zero
  - Recovery is zero (LGD is 100%)
- Granularity very low for $n = 1$, very high for $n = 1000$
  - Expected loss then equals default probability

Granularity reduces risk: results of the example

- Portfolio with low granularity *or* very low default probability
  - Has binary "0-1" risk, behaves like a coin toss
  - High probability of no loss, small but material probability π of complete loss
- Portfolio with high granularity
  - Very narrow range of likely loss rates
  - Loss rate very likely to be close to expected loss/default probability π
- Portfolio with moderate granularity
  - Has a wider range of likely outcomes
  - And thus greater probability of material unexpected loss
  - Has low probability of complete or near-complete loss
  - Impact of granularity greater for higher default probability π

Credit VaR, granularity, and default probability



Cumulative probability distribution function of losses for n equally-sized loans and default probabilities π, as a fraction of portfolio value. The x-axis measures the loss rate in the portfolio as a fraction of the portfolio's total par value; the y-axis measures probability. Each point shows the probability of a realized loss of that magnitude or less. Cyan grid line placed at 99 percent credit VaR. Orange grid line placed at expected loss and is the same in each column.

Credit loss distribution in an uncorrelated portfolio

No diversification: For $n = 1$

$$\text{credit VaR} = \begin{Bmatrix} -\text{EL} \\ 1 - \text{EL} \end{Bmatrix} \quad \text{for} \quad \pi \begin{Bmatrix} < \\ \geq \end{Bmatrix} 1 - \alpha$$

High granularity: credit VaR$\rightarrow 0$ as $n \rightarrow \infty$

n		$\pi = 0.005$	$\pi = 0.025$	$\pi = 0.05$
1	0.99-quantile of credit losses	0.00000	1.00000	1.00000
	Credit VaR at 99% confidence	-0.00500	0.97500	0.95000
50	0.99-quantile of credit losses	0.04000	0.08000	0.14000
	Credit VaR at 99% confidence	0.03500	0.05500	0.09000
1000	0.99-quantile of credit losses	0.01100	0.03700	0.06700
	Credit VaR at 99% confidence	0.00600	0.01200	0.01700
50000	0.99-quantile of credit losses	0.00574	0.02664	0.05228
	Credit VaR at 99% confidence	0.00074	0.00164	0.00228

Expressed as a fraction of portfolio par value.

Granularity and coherence

- Low granularity—even with low default correlation—is associated with
  - Negative credit VaR
  - And violations of coherence of VaR, esp. subadditivity property
- **Example**: single credit with $R = 0$, default probability π
  - For VaR confidence level α, portfolio VaR will be negative for

$$\pi < 1 - \sqrt{\alpha}$$

Violation of subadditivity

- Concentrated portfolios may have higher credit VaR than constituent securities
- **Example**: equal amounts of two identical uncorrelated credits
  - Loss distribution with $\rho_{12} = 0$, $R = 0$, default probability π

Both default	π^2
1 default	$2(\pi - \pi^2) = 2\pi(1-\pi)^2$
No default	$(1-\pi)^2 = 1 - 2\pi + \pi^2$

- For any VaR confidence level α, portfolio VaR will be negative for

$$(1-\pi)^2 > \alpha \quad \leftrightarrow \quad \pi < 1 - \sqrt{\alpha}$$

  - E.g. for $\alpha = 0.99, \pi < 1 - \sqrt{0.99} = 0.0050126$
- Violates subadditivity property of coherence for $1 - \sqrt{\alpha} \leq \pi < 1 - \alpha$

Loss (%)	$\pi = 0.005$	$\pi = 0.00525$
0.0	0.990025	0.989528
0.5	0.00995	0.0104449
1.0	0.000025	0.0000275625

- Provides incentive in VaR-based limit system for separating low probability/high loss credits into distinct portfolios

What problem does the copula approach solve?

- Factor models make many assumptions
  - Structural model, need to identify factors correctly
  - Little role for idiosyncratic risk
- $\rightarrow$Search for models with market-informed parameters
  - Useful for estimating spread risk of portfolio credit products
- A **copula** is a postulated parametric family of joint distributions
  - Exploits the little information we have on portfolio default distribution
  - Choice of copula a judgement call
  - Trade-off between ability to capture tail risk and need to estimate/guess at additional parameters
- Facilitates estimation of joint distribution via simulation

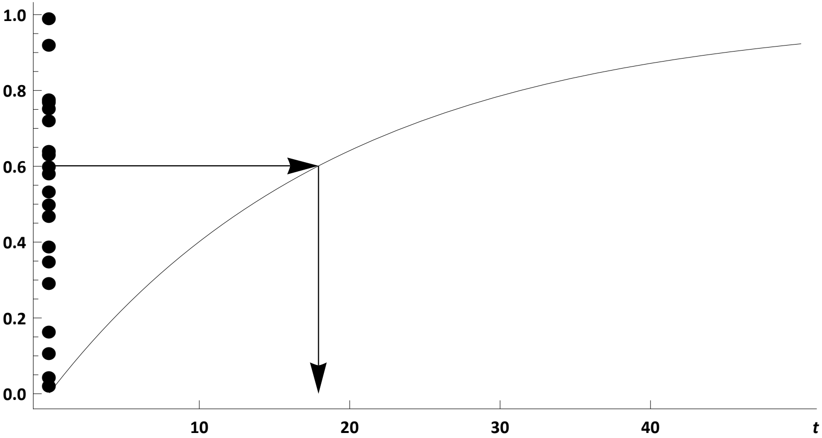
Information needed to apply copula approach

- Default distribution of each individual single credit
  - We have some information: default probabilities from ratings, credit spreads
- Default correlations
  - We have some information from estimates of asset or equity return correlations, implied correlations from equity and credit derivatives
  - But much less knowledge than of default probabilities
  - May need to assume all default correlations identical, estimate "general level"
- Little else known about the joint distribution of credit losses

Sketch of the procedure

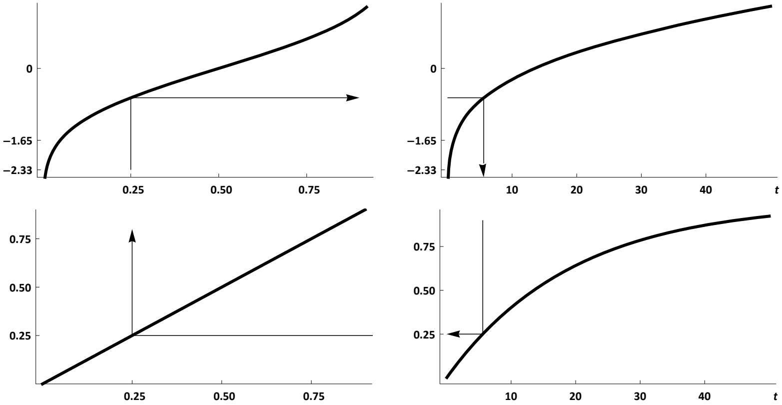
- Generate simulations from chosen copula, e.g. multivariate standard normal with specified correlation matrix
- Map each simulated value into a value of the associated cumulative probability distribution function
  - For example, a simulated standard normal variate equal to 1.0 maps to a probability of 83.13 percent, -2.33 to a probability of 1.0 percent
  - Copula approach assumes these standard normals rather than defaults are jointly normally distributed
- Use the default time distributions of individual credits to map from a probability to a simulated default time

Simulating single-credit default times



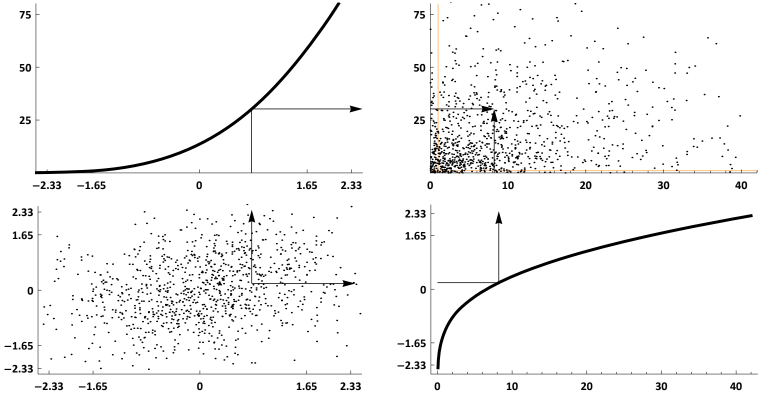
Cumulative default time distribution for a credit with a one-year default probability of 0.05⇒hazard rate is 0.0513. Points represent 20 simulated values of the uniform distribution.

Shifting from uniform to normal simulations



Graph traces how to change one thread of a uniform simulation to a normal simulation. The lower right panel shows the default time distribution for a credit with a one-year default probability of 5 percent.

Simulating multiple defaults



Lower left quadrant displays 1000 simulations from a bivariate standard normal with a correlation coefficient of 0.25. The lower right (upper left) panel shows the default time distribution for a credit with a one-year default probability of 10 percent (5 percent), with cumulative probabilities expressed as standard normal quantiles. Orange grid lines in the upper right quadrant partition the simulation results into default times less than and greater than one year for each obligor.